RESOLUTION OF DG-MODULES

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Abstract. Recently, Yekutieli introduced projective dimension, injective dimension and flat dimension of DG-modules by generalizing the characterization of projective dimension and injective dimension of ordinary modules by vanishing of Ext-group. In this paper, we introduce DG-version of projective resolution and injective resolution for DG-modules over a connective DG-algebra which are different from known DG-version of projective, injective and flat resolutions. An important feature of these resolutions is that, roughly speaking, the "length" of these resolutions give projective, injective or flat dimensions. We show that these resolutions allows us to investigate basic properties of projective and injective dimensions of DG-modules. As an application we introduce the global dimension of a connective DG-algebra and show that finiteness of global dimension is derived invariant.

1. Introduction

Differential graded (DG) algebra lies in the center of homological algebra and allows us to use techniques of homological algebra of ordinary algebras in much wider context. The projective resolutions and the injective resolutions which are the fundamental tools of homological algebra already have their DG-versions, which are called a DG-projective resolution and a DG-injective resolution. The aim of this paper is to introduce a different DG-versions for DG-modules over a connective DG-algebra. The motivation came from the projective dimensions and the injective dimensions for DG-modules introduced by Yekutieli.

We explain the details by focusing on the projective dimension and the projective resolution. Let R be an ordinary algebra. One of the most fundamental and basic homological invariant for a (right) R-module M is the projective dimension $\operatorname{pd}_R M$. Avramov-Foxby [1] generalized the projective dimension for an object of the derived category $M \in \mathsf{D}(R)$. Recently, Yekutieli [3] introduced the projective dimension $\operatorname{pd}_R M$ for an object of $M \in \mathsf{D}(R)$ in the case where R is a DG-algebra from the view point that the number $\operatorname{pd}_R M$ measures how the functor $\mathbb{R}\mathrm{Hom}_R(M, -)$ changes the amplitude of the cohomology groups.

Let R be an ordinary ring and M an R-module, again. Recall that the projective dimension $\operatorname{pd}_R M$ is characterized as the smallest length of projective resolutions $P_\bullet$

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

There are the notion of DG-projective resolution, which is also called projectively cofibrant replacement and so on, which is a generalization of a projective resolution for a DG-module M over a DG-algebra R. However, it is not suitable to measure the projective dimension. The aim of this paper is to introduce a notion of a sup-projective (sppj) resolution of an

The detailed version of this paper is posted on arXiv:1802.01994 and will be submitted for publication elsewhere.

object of $M \in \mathsf{D}(R)$ which can measure the projective dimension, in the case where R is a connective (=non-positive) DG-algebra.

Recall that a (cohomological) DG-algebra R is called *connective* if the vanishing condition $\mathrm{H}^{>0}(R) = 0$ of the cohomology groups is satisfied. There are rich sources of connective DG-algebras: the Koszul algebra $K_R(x_1, \cdots, x_d)$ in commutative ring theory, and an endomorphism DG-algebra $\mathbb{R}\mathrm{Hom}(S, S)$ of a silting object S. We would like to point out that a commutative connective DG-algebras are regarded as the coordinate algebras of derived affine schemes in derived algebraic geometry (see e.g. [2]).

Let R be a connective DG-algebra. We set $\mathcal{P} := \mathrm{Add}\, R \subset \mathsf{D}(R)$ to be the additive closure of R inside $\mathsf{D}(R)$. Namely, $\mathcal{P}$ is the full subcategory consisting of $M \in \mathsf{D}(R)$ which is a direct summand of some coproduct of R. In sppj resolution, $\mathcal{P}$ plays the role of projective modules in the usual projective resolution.

A sppj resolution $P_\bullet$ of $M \in \mathsf{D}^{<\infty}(R)$ is a sequence of exact triangles $\{\mathcal{E}_i\}_{i \geq 0}$

$$\mathcal{E}_i : M_{i+1} \xrightarrow{g_{i+1}} P_i \xrightarrow{f_i} M_i \to$$

such that f_i is a sppj morphism (Definition 6), where we set $M_0 := M$. We often exhibit a sppj resolution $P_\bullet$ as below by splicing $\{\mathcal{E}_i\}_{i \geq 0}$

$$P_\bullet : \cdots \to P_i \xrightarrow{\delta_i} P_{i-1} \xrightarrow{\delta_{i-1}} \cdots \to P_1 \xrightarrow{\delta_1} P_0 \to M$$

where we set $\delta_i := g_i f_i$. It is analogous to that in the case where R is an ordinary algebra, a projective resolution $P_\bullet$ of an R-module M is constructed by splicing exact sequences

$$0 \to M_{i+1} \to P_i \to M_i \to 0$$

with P_i projective.

We state the main result which gives equivalent conditions of $\mathrm{pd}\, M = d$.

Theorem 1 (Theorem 10)**.** *Let $M \in \mathsf{D}^{<\infty}(R)$ and $d \in \mathbb{N}$ a natural number. Then the following conditions are equivalent*

(1) $\mathrm{pd}\, M = d$.
(2) *For any sppj resolution $P_\bullet$, there exists a natural number $e \in \mathbb{N}$ which satisfying the following properties*
  (a) *M_e belongs to $\mathcal{P}[-\sup M_e]$.*
  (b) *$d = e + \sup P_0 - \sup M_e$.*
  (c) *The structure morphism $g_e : M_e \to P_{e-1}$ is not a split-monomorphism.*
(3) *M has sppj resolution $P_\bullet$ of length e which satisfies the following properties.*

$$P_e \xrightarrow{\delta_e} P_{e-1} \xrightarrow{\delta_{e-1}} \cdots P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{f_0} M$$

  (a) *$d = e + \sup P_0 - \sup P_e$.*
  (b) *The e-th differential δ_e is not a split-monomorphism.*
(4) *The functor $F = \mathbb{R}\mathrm{Hom}(M, -)$ sends the standard heart* $\mathrm{Mod}\, \mathrm{H}^0(R)$ *to* $\mathsf{D}^{[-\sup M, d-\sup M]}(R)$ *and there exists $N \in \mathrm{Mod}\, H^0$ such that $\mathrm{H}^{d-\sup M}(F(N)) \neq 0$.*
(5) *d is the smallest number which satisfies*

$$M \in \mathcal{P}[-\sup M] * \mathcal{P}[-\sup M + 1] * \cdots * \mathcal{P}[-\sup M + d].$$

The condition (4) tells that the projective dimension of M can be measured by only looking the standard heart $\operatorname{Mod} \mathrm{H}^0(R)$ of the derived category $\mathsf{D}(R)$. The condition (5) says that the projective dimension $\operatorname{pd} M$ is the smallest number of extensions by which we obtain M from the "projective objects" $\mathcal{P}$ (see Definition 5).

We introduce the global dimension $\operatorname{gldim} R$ of a connective DG-algebra R. For an ordinary ring R, a key result to define the global dimension $\operatorname{gldim} R$ is that the supremum of the projective dimensions $\operatorname{pd} M$ of all R-modules M and that of the injective dimensions $\operatorname{injdim} M$ coincide. We provide a similar result for a connective DG-algebra R. It is well-known that the ordinary global dimensions is not preserved by derived equivalence, but their finiteness is preserved. We prove the DG-version of this result.

1.1. **Notation and convention.** The basic setup and notations are the followings.

Throughout the paper, we fix a base commutative ring $\mathbf{k}$ and (DG, graded) algebra is (DG, graded) algebra over $\mathbf{k}$. We denote by $R = (R, \partial)$ a connective cohomological DG-algebra. Recall that "connective" means that $\mathrm{H}^{>0}(R) = 0$. We note that every connective DG-algebra R is quasi-isomorphic to a DG-algebra S such that $S^{>0} = 0$. Since quasi-isomorphic DG-algebras have equivalent derived categories, it is harmless to assume that $R^{>0} = 0$ for our purpose.

For notational simplicity we set $H := \mathrm{H}(R)$ and $H^0 := \mathrm{H}^0(R)$.

We denotes by $\mathsf{C}(R)$ the category of DG-R-modules and cochain morphisms, by $\mathsf{K}(R)$ the homotopy category of DG-R-modules and by $\mathsf{D}(R)$ the derived category of DG R-modules. The symbol Hom denotes the Hom-space of $\mathsf{D}(R)$.

Let $n \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$. The symbols $\mathsf{D}^{<n}(R)$, $\mathsf{D}^{>n}(R)$ denote the full subcategories of $\mathsf{D}(R)$ consisting of M such that $\mathrm{H}^{\geq n}(M) = 0$, $\mathrm{H}^{\leq n}(M) = 0$ respectively. We set $\mathsf{D}^{[a,b]}(R) = \mathsf{D}^{\geq a}(R) \cap \mathsf{D}^{\leq b}(R)$ for $a, b \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ such that $a \leq b$. We set $\mathsf{D}^{\mathrm{b}}(R) := \mathsf{D}^{<\infty}(R) \cap \mathsf{D}^{>-\infty}(R)$.

2. Projective dimension of DG-modules and sppj resolution

2.1. **Projective dimension of $M \in \mathsf{D}(R)$ after Yekutieli.** We recall the definition of the projective dimension of $M \in \mathsf{D}(R)$ introduced by Yekutieli.

Definition 2 ([3, Definition 2.4]). Let $a \leq b \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$.

(1) An object $M \in \mathsf{D}(R)$ is said to have *projective concentration* $[a, b]$ if the functor $F = \mathbb{R}\mathrm{Hom}_R(M, -)$ sends $\mathsf{D}^{[m,n]}(R)$ to $\mathsf{D}^{[m-b,n-a]}(\mathbf{k})$ for any $m \leq n \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$.
$$F(\mathsf{D}^{[m,n]}(R)) \subset \mathsf{D}^{[m-b,n-a]}(\mathbf{k}).$$
(2) An object $M \in \mathsf{D}(R)$ is said to have *strict projective concentration* $[a, b]$ if it has projective concentration $[a, b]$ and does't have projective concentration $[c, d]$ such that $[c, d] \subsetneq [a, b]$.
(3) An object $M \in \mathsf{D}(R)$ is said to have projective dimension $d \in \mathbb{N}$ if it has strict projective concentration $[a, b]$ for $a, b \in \mathbb{Z}$. such that $d = b - a$.

In the case where, M does't have a finite interval as projective concentration, it is said to have infinite projective dimension.

We denote the projective dimension by $\operatorname{pd} M$.

The following observations are useful.

Lemma 3. *If $M \in \mathsf{D}(R)$ has finite projective dimension, then it belongs to $\mathsf{D}^{<\infty}(R)$.*

Lemma 4. *An object $M \in \mathsf{D}^{<\infty}(R)$ has projective dimension d if and only if it has strict projective concentration $[\sup M - d, \sup M]$.*

2.2. **The class $\mathcal{P}$ and sup-projective (sppj) resolution.** We introduce a class $\mathcal{P}$ of DG-modules which plays a role of the class of projective modules.

Definition 5. We denote by $\mathcal{P} \subset \mathsf{D}(R)$ the full subcategory of direct summands of a direct sums of R. In other words, $\mathcal{P} = \operatorname{Add} R$.

We give the definition of a sup-projective (sppj) resolution of $M \in \mathsf{D}^{<\infty}(R)$.

Definition 6 (sppj morphism and sppj resolution)**.** Let $M \in \mathsf{D}^{<\infty}(R), M \neq 0$.

(1) A sppj morphism $f : P \to M$ is a morphism in $\mathsf{D}(R)$ such that $P \in \mathcal{P}[-\sup M]$ and the morphism $\mathrm{H}^{\sup M}(f)$ is surjecitve.
(2) A sppj morphism $f : P \to M$ is called minimal if the morphism $\mathrm{H}^{\sup M}(f)$ is a projective cover.
(3) A sppj resolution $P_\bullet$ of M is a sequence of exact triangles for $i \geq 0$ with $M_0 := M$

$$M_{i+1} \xrightarrow{g_{i+1}} P_i \xrightarrow{f_i} M_i$$

such that f_i is sppj.

The following inequality folds

$$\sup M_{i+1} = \sup P_{i+1} \leq \sup P_i = \sup M_i.$$

For a sppj resolution $P_\bullet$ with the above notations, we set $\delta_i := g_{i-1} \circ f_i$.

$$\delta_i : P_i \to P_{i-1}.$$

Moreover we write

$$\cdots \to P_i \xrightarrow{\delta_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{\delta_1} P_0 \to M.$$

(4) A sppj resolution $P_\bullet$ is said to have length e if $P_i = 0$ for $i > e$ and $P_e \neq 0$.
(5) A sppj resolution $P_\bullet$ is called minimal if f_i is minimal for $i \geq 0$.

The following two lemmas give a motivation to introduce sppj resolutions.

Lemma 7. *Let $M \in \mathsf{D}^{<\infty}(R)$ and $f : P \to M$ a sppj morphism and $N := \operatorname{cone}(f)[-1]$ the cocone of f. Then the following assertions hold.*

(1) *Assume that $1 \leq \operatorname{pd} M$. Then, $\operatorname{pd} N = \operatorname{pd} M - 1 - \sup M + \sup N$.*
(2) *Assume that $\operatorname{pd} M = 0$. Then f is a split-epi morphism and hence M is a direct summand of P.*

An important consequence is the following.

Corollary 8. *Let $M \in \mathsf{D}(R) \setminus \{0\}$. Then $\operatorname{pd} M = 0$ if and only if $M \in \mathcal{P}[-\sup M]$.*

2.3. **Projective dimension and the length of sppj-resolutions.**

Theorem 9. *Let $M \in \mathsf{D}^{<\infty}(R)$ and $d \in \mathbb{N}$ a natural number. Then, the following conditions are equivalent*

(1) $\operatorname{pd} M \leq d$.
(2) *For any sppj resolution $P_\bullet$, there exists a natural number $e \in \mathbb{N}$ such that $M_e \in \mathcal{P}[-\sup P_e]$ and $e + \sup P_0 - \sup M_e \leq d$. In particular, we have a sppj resolution of length e.*
$$M_e \to P_{e-1} \to P_{e-2} \to \cdots P_1 \to P_0 \to M.$$
(3) *M has sppj resolution $P_\bullet$ of length e such that $e + \sup P_0 - \sup P_e \leq d$.*
(4) *The functor $F = \mathbb{R}\mathrm{Hom}(M, -)$ sends the standard heart* $\operatorname{Mod} H^0$ *to* $\mathsf{D}^{[-\sup M, d-\sup M]}(R)$.
(5) *M belongs to $\mathcal{P}[-\sup M] * \mathcal{P}[-\sup M + 1] * \cdots * \mathcal{P}[-\sup M + d]$.*

Theorem 10. *Let $M \in \mathsf{D}^{<\infty}(R)$ and $d \in \mathbb{N}$ a natural number. Then the following conditions are equivalent*

(1) $\operatorname{pd} M = d$.
(2) *For any sppj resolution $P_\bullet$, there exists a natural number $e \in \mathbb{N}$ which satisfying the following properties*
  (a) $M_e \in \mathcal{P}[-\sup M_e]$.
  (b) $d = e + \sup P_0 - \sup M_e$.
  (c) *g_e is not a split-monomorphism.*
(3) *M has sppj resolution $P_\bullet$ of length e which satisfies the following properties.*
  (a) $d = e + \sup P_0 - \sup P_e$.
  (b) *δ_e is not a split-monomorphism.*
(4) *The functor $F = \mathbb{R}\mathrm{Hom}(M, -)$ sends the standard heart* $\operatorname{Mod} H^0$ *to* $\mathsf{D}^{[-\sup M, d-\sup M]}(R)$ *and there exists $N \in \operatorname{Mod} H^0$ such that $\mathrm{H}^{d-\sup M}(F(N)) \neq 0$.*
(5) *d is the smallest number which satisfies*
$$M \in \mathcal{P}[-\sup M] * \mathcal{P}[-\sup M + 1] * \cdots * \mathcal{P}[-\sup M + d].$$

Remark 11. Injective dimension $\operatorname{injdim} M$ of DG-module M was defined in a similar way in [3]. Inf-injective (ifij) resolution is defined in a similar way. Almost all properties can be proved in dual ways of projective dimensions and sppj-resolution. However to introduce the class $\mathcal{I}$, which is the counterpart of the class $\mathcal{P}$, we need to work.

3. Global dimension

We introduce the notion of the global dimension of a connective DG-algebra R.

Theorem 12. *Let R be a connective DG-algebra. Then the following numbers are the same.*

(1) $\sup\{\operatorname{pd} M - \operatorname{amp} M \mid M \in \mathsf{D}^{<\infty}(R)\}$
(2) $\sup\{\operatorname{pd} M \mid M \in \operatorname{Mod} H^0\}$
(3) $\sup\{\operatorname{injdim} M - \operatorname{amp} M \mid M \in \mathsf{D}^{>-\infty}(R)\}$
(4) $\sup\{\operatorname{injdim} M \mid M \in \operatorname{Mod} H^0\}$

This common number is called the (right) global dimension *of R and is denoted as* $\operatorname{gldim} R$.

We point out the following

Remark 13. For any connective DG-algebra, we have

$$\sup\{\operatorname{pd} M \mid M \in \mathsf{D}^{<\infty}\} = \infty,$$

since $\operatorname{pd}(R \oplus R[n]) = n$ for $n \in \mathbb{N}$.

Observe that if R is an ordinary algebra, then the global dimension defined in Theorem 12 coincides with the ordinary global dimension. The ordinary global dimensions is not preserved by derived equivalence, but their finiteness is preserved. We prove the DG-version.

Let R and S be connective DG-algebra. Assume that they are derived equivalent to each other. Namely, there exists an equivalence $\mathsf{D}(R) \simeq \mathsf{D}(S)$ of triangulated categories, by which we identify $\mathsf{D}(R)$ with $\mathsf{D}(S)$.

Proposition 14. *Under the above situation the following assertions hold.*

(1) $\operatorname{pd}_S R < \infty$.
(2) $\operatorname{gldim} S \leq \operatorname{gldim} R + \operatorname{pd}_S R$.
(3) $\operatorname{gldim} R < \infty$ *if and only if* $\operatorname{gldim} S < \infty$.

At the end of the proceeding, we give an answer to one of the question from Prof. Kikumasa in the symposium.

Proposition 15. *For a connective DG-algebra R, the following conditions are equivalent.*

(1) $\operatorname{gldim} R = 0$.
(2) *R is an ordinary algebra (i.e., $H^{<0} = 0$) which is semi-simple.*

References

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